# Efficiency

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# Recap

- We saw last time that any standard of social welfare is problematic in a precise sense.
- If we want to proceed, we need to compromise in some way.
- We must abandon one of the basic principles
  1. Universal Domain
  2. Pareto
  3. Independence of Irrelevant Alternatives

# Pareto

1. Pareto is the criterion most closely tied to social welfare.
2. So we will insist on Pareto
3. What if we *only* require Pareto?

# Pareto Dominance

**Definition**

Alternative $A$ Pareto dominates another alternative $B$ if every individual prefers $A$ to $B$, i.e. $A \succ_i B$ for every individual $i$.

1. Pareto dominance is a way of ranking alternatives.
2. But it is an *incomplete* ranking: often neither alternative Pareto dominates the other.
3. Examples:
    1. The last remaining basketball ticket.
    2. Public school assignment.
    3. Designer dress dibs.

# Pareto Efficiency

- So Pareto dominance rarely gives us a clear ranking
- But when it does, the prescription couldn't be more compelling.

**Definition**

An alternative $A$ is *Pareto efficient* if there is no $B$ that Pareto dominates it.

- We should not choose any alternative which is Pareto dominated.
- This is a foundational principle of Economics.
- Unfortunately that still leaves us with a lot of alternatives and no way to compare them.

# But Wait

1. Let's revisit the example with the basketball ticket.
2. Let's suppose we also have the possibility of enforcing monetary transfers.
3. How much money are you willing to pay to have the ticket?

# Willingness to Pay

Thought experiment.

- Pile of money.
- Basketball ticket.

How large can we make the pile of money before you take the money rather than fly?

We equate that with your *willingness to pay.*

# Willingness to Pay

- Willingness to pay adds more information about your preferences.
- Before we just talked about your ranking of $A$ versus $B$.
- Now we can say something about *how much* more you like $A$ than $B$.
- How much money would it take to get you to favor $B$ over $A$?
- Truthfully.

# Pareto Dominance When Money's Involved

- Remember that any allocation of the ticket is Pareto efficient.
- Suppose we are going to give the ticket to $j$ but $i$ has a higher willingness to pay.
- Consider now the following new alternative.
  1. We give the ticket to $i$ instead of $j$.
  2. We take an amount of money $x$ from $i$ and transfer it to $j$.
  3. $x$ is chosen to be *in between* the (high) willingness to pay of $i$ and the (low) willingness to pay of $j$.
- This alternative Pareto dominates giving the ticket to $j$ (and no exchange of money.)

# More Generally

## Proposition

*When money is involved, the only Pareto efficient alternative is to give the ticket to the fan with the highest willingness to pay.*

- Consider giving the ticket to a fan with a lower willingness to pay.
- We just saw how to construct a Pareto dominating alternative/monetary transfer.
- If it's Pareto dominated then it's not Pareto efficient.

# Money, Formally Now

- We will assume that *monetary transfers* are possible and can be enforced.
- A monetary transfer scheme can be represented by $t = (t_1, \ldots, t_n)$ where
  - $t_i$ denotes the amount of money paid by individual $i$. (could be negative, a subsidy)
  - $\sum_{i=1}^{n} t_i = t_1 + t_2 + \ldots + t_n$ is the *budget surplus.* (could be negative, a deficit)
  - $\sum_{i=1}^{n} t_i = 0$ means that the transfer scheme has a *balanced budget.*

# Social Choices with Monetary Transfers

- Remember that society must choose an alternative.
- Now alternatives have two components.
  - A choice from $\mathcal{A}$ (e.g. who gets the ticket and who doesn't)
  - A monetary transfer scheme $t$ (i.e. who pays, who gets paid, and how much.)
- And now we must describe the individuals' preferences over both components. (i.e. how do they trade-off monetary payments versus better/worse alternatives.)

# Money Utility

Willingness to pay is captured by utility functions.

## Definition

The *value* to individual $i$ from alternative $x$ is denoted $v_i(x)$. The *utility* associated with alternative $x$ together with monetary transfer $t_i$ is

$$U_i(x, t_i) = v_i(x) - t_i$$

Individual $i$ prefers a pair $(x, t_i)$ to a pair $(y, t_i')$ if $U_i(x, t_i) \geq U_i(y, t_i')$ and if the inequality is strict, we say his preference is *strict.*

As always in economics, a utility function is just a mathematical device that allows us to describe preferences in a precise way.

Let's verify that a utility function like $U_i$ describes wilingness to pay.

# Money Utility and WTP

## Example

Suppose there is one ticket left. Alternative $A$ is you get it, alternative $B$ is I get it. Suppose that you derive no value from *me* seeing the game, so $v_{\text{you}}(B) = 0$ and that your value from seeing the game is $v_{\text{you}}(A)$ (some positive number.) If you are asked to choose between having the ticket ($A$) and paying $t_{\text{you}}$ dollars versus not seeing the game ($B$) and paying nothing, you would be willing to pay whenever

$$U_{\text{you}}(A, t_{\text{you}}) \geq U_{\text{you}}(B, 0)$$

which translates to

$$v_{\text{you}}(A) - t_{\text{you}} \geq 0$$

or

$$t_{\text{you}} \leq v_{\text{you}}(A)$$

This says that you are willing to pay (up to but no more than) $v_{\text{you}}(A)$ to see the game.

# More on WTP

More generally, if $A$ and $B$ are any two alternatives, and $t_i$ is a number, individual $i$ prefers $(A, t)$ to $(B, 0)$ whenever

$$U_i(A, t_i) \geq U_i(B, 0)$$

which translates to

$$t_i \leq v_i(A) - v_i(B)$$

so that $v_i(A) - v_i(B)$ measures $i$'s willingness to pay to have $A$ rather than $B$. (And this may be negative.)

# Maximizing Social Value

1. Recall the allocation of the ticket.
2. Pareto efficiency implied giving it to the fan with the highest willingness to pay.
3. In fact that's the alternative that maximizes the total value in society.
4. That was a special problem
   - You have positive value for the one alternative where you get the ticket.
   - You have zero value for everything else.
5. In typical problems you will have different, non-zero values for many different alternatives.
   - School assignment
   - Ad placement
   - etc.

# Maximizing Social Value

Still, we are lead to consider the alternative $A$ that maximizes total value:

$$\sum_i v_i(A)$$

- This is called the *utilitarian* alternative.
- Just as in the simple ticket example, the utilitarian alternative is the only Pareto efficient alternative when monetary transfers are possible.

# Utilitarianism and Pareto efficiency

Let $A$ be the utilitarian alternative and $B$ be any other alternative. Then

$$\sum_i v_i(A) > \sum_i v_i(B)$$

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(Note that this is positive for those who like $A$ better than $B$, negative otherwise.)

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(Note that this is positive for those who like $A$ better than $B$, negative otherwise.)
Everyone is indifferent between $(A, t)$ and $B$.

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# Utilitarianism and Pareto efficiency

But notice that $\hat{t}$ has a budget surplus:

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And because $A$ is utilitarian, this is positive. We can now construct a new transfer scheme $t$ by reducing each $\hat{t}_i$ by a small amount, balancing the budget and making everybody strictly better off.

# The Utilitarian Social Welfare Function

With willingness to pay as a measure of preference, we can now define a social welfare function which utilizes that information.

**Definition**

Under the utilitarian social welfare function, society prefers $(A, t)$ to $(B, t')$ if $\sum_{i=1}^{n} U_i(A, t_i) \geq \sum_{i=1}^{n} U_i(B, t'_i)$. In particular, if $t$ and $t'$ have balanced budgets then this reduces to

$$\sum_{i=1}^{n} v_i(A) \geq \sum_{i=1}^{n} v_i(B)$$

This social welfare function satisfies IIA and Pareto and is not a dictatorship.

# Not Perfect

- Willingness to accept vs. willingness to pay. (and ability to pay.)
- Arguably not comparable across people.
- Time rather than money?

# Pareto Efficiency Again

For the remainder of this lecture, we restrict attention to monetary transfer schemes that have a balanced budget.

## Definition

Social choice $(A, t)$ Pareto dominates another choice $(B, t')$ if every individual prefers $(A, t)$ to $(B, t')$ and at least one individual strictly prefers it.

## Definition

A social choice $(A, t)$ is *Pareto efficient* if there is no $(B, t')$ that Pareto dominates it.

# Utilitarianism and Pareto Efficiency

As we have shown, Pareto efficiency implies utilitarianism.

**Proposition**

*When monetary transfers are possible, if $(A, t)$ is Pareto efficient, then $A$ must be utilitarian as well.*

# Utilitarianism and Pareto efficiency

The converse is true too.

**Proposition**

*When monetary transfers are possible, if $A$ is utilitarian and $t$ is a budget-balanced transfer scheme, then $(A, t)$ is Pareto efficient.*

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Suppose $A$ is utilitarian. Suppose there was a $(B, \hat{t})$ that would Pareto dominate $(A, t)$. That would mean

$$v_i(B) - \hat{t}_i \geq v_i(A) - t_i$$

for all $i$ with at least one strict inequality. Summing over $i$

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