Incentives and Game Theory

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Example

Suppose that you and your roommate are considering buying an espresso machine for your apartment. The machine costs $50. Your willingness to pay is $v_1 = 40$. You know that your roommate has a willingness to pay $v_2$ but you don’t know what it is. (Your values are zero when you do not purchase the machine.)

You and your roommate are choosing among the following alternatives.

1. no machine, no monetary payments.
2. espresso machine, any transfer scheme $t = t_1, t_2$ such that $t_1 + t_2 = 50$. 

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Utilitarianism in Your Apartment

The utilitarian policy is to purchase the espresso machine if

$$v_1 + v_2 \geq 50$$

and not to purchase the machine if $v_1 + v_2 < 50$. 
\[ v_1 + v_2 = 50 \]
Purchase the machine

\[ v_1 + v_2 = 50 \]

\[ \text{don't} \]
Split the Cost?

- Suppose you agree to split the cost.
- When would you be willing to do it?
- Only when $v_1 \geq 25$.
- And the same is true of your roommate.
Split the Cost is Inefficient

\[ v_2 \]

\[ 25 \]

\[ v_1 \]

you agree
Split the Cost is Inefficient
Split the Cost is Inefficient

Purchase the machine

$v_2$

25

$v_1$

25
Split the Cost is Inefficient

The diagram illustrates a decision point where two parties, labeled $v_1$ and $v_2$, are considering purchasing a machine. The diagram shows two scenarios:

- **Purchase the machine**: The area above the line where $v_1 = 25$ and $v_2 = 25$ indicates the condition under which both parties should purchase the machine.
- **should purchase but don’t**: The area below the line indicates that one party should purchase the machine but does not, leading to an inefficient outcome.

This scenario highlights the inefficiency of splitting the cost in certain conditions.
Can we devise a mechanism which satisfies the following two conditions

1. Gets the two to tell truths about values
2. Allows to microwave to be bought whenever it should
Contribution Game

- The roommates simultaneously pledge a contribution (some number.)
- If the contributions add up to at least 50 then the espresso machine is bought (and the surplus divided proportionally)
- Otherwise not.
A game is described by

1. The players \( i = 1, \ldots, n \),
2. The choices available to each of them: \( A_i \). (called the *actions* or *strategies*.)
   - Player \( i \) chooses one \( a_i \) from \( A_i \).
   - The players choose simultaneously.
3. The *outcomes*: \( a = (a_1, a_2, \ldots, a_n) \) where \( a_i \in A_i \) for each \( i \).
   - When we write \( a_{-i} \), we mean \( (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \).
   - Then \( (a_i, a_{-i}) \) is another way of writing \( a \).
4. *Utilities*: \( \pi_i(a) \) (also called *payoffs*.)
   - \( \pi_i(a) \) is a number representing the preference for outcome \( a \).
   - Player \( i \) wants to choose his action \( a_i \) to maximize \( \pi(a_i, a_{-i}) \).
   - But player \( i \) has no control over \( a_{-i} \).
Golden Balls

Yes indeed. Golden Balls
Payoff Matrix

<table>
<thead>
<tr>
<th></th>
<th>split</th>
<th>steal</th>
</tr>
</thead>
<tbody>
<tr>
<td>split</td>
<td>50, 50</td>
<td>0, 100</td>
</tr>
<tr>
<td>steal</td>
<td>100, 0</td>
<td>0, 0</td>
</tr>
</tbody>
</table>
Dominated Strategies

Definition

Strategy $a_i$ dominates strategy $a_i'$ if $a_i$ always gives a payoff at least as high as $a_i'$ and sometimes strictly higher. In formal terms, $a_i$ dominates $a_i'$ if

1. for every action profile $a_{-i}$ of the opponents,

$$\pi_i(a_i, a_{-i}) \geq \pi_i(a_i', a_{-i})$$

2. and for at least one action profile $\hat{a}_{-i}$ of the opponents,

$$\pi_i(a_i, \hat{a}_{-i}) > \pi_i(a_i', \hat{a}_{-i})$$
Dominant Strategies

Definition
A strategy $a_i$ is dominant if it dominates all other strategies.
Super Golden Balls

- Each player has 100 pairs of balls.
- The game goes for up to 100 rounds.
- Each round an additional $100,000 at stake.
- Each time they both say split, they each earn $50,000.
- The first round in which either of them say steal,
  - The game ends.
  - If only one person said steal, that person gets the whole $100,000 from that round.
- If neither chooses steal for 100 rounds the game ends. (And by then they have won $5,000,000.)
Iterative Removal of Dominated Strategies

- We begin by removing from consideration the dominated strategies.
- We consider the reduced game that remains.
- Now we remove strategies that are dominated in the reduced game.
- We continue with this until there is nothing more to remove.
Generous Balls

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<td>1,0</td>
</tr>
</tbody>
</table>
Not Every Game Has Dominated Strategies

\[
\begin{array}{c|cc}
& \text{red} & \text{blue} \\
\hline
\text{red} & -10, -10 & 1, 0 \\
\text{blue} & 0, 1 & -10, -10 \\
\end{array}
\]

\textbf{Figure:} The Dress Dibs Game

What about the contribution game from earlier?