The Vickrey-Clarke-Groves Mechanism

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Dealing with Externalities

- We saw that the Vickrey auction was no longer efficient when there are externalities.
- But we can modify the rules to restore efficiency.
- Recall the example from last time:

	X	Y	Z
x	v_x	0	0
y	0	v_y	-5
z	0	0	v_z

- Modified auction:
  - Subtract 5 from z's bid. Set $\hat{b}_z = b_z - 5$
  - Award the object to the highest bidder where we use $\hat{b}_z$ for z.
  - If x or y win, they pay the highest losing bid, again using $\hat{b}_z$.
  - If z wins, she pays the highest losing bid *plus 5*.

More examples

- But what if we don't know the level of the externality?
- And what about other problems? The designer dress problem?

	Blue	Red
Chris	v_c(blue)	v_c(Red)
Pat	v_p(blue)	v_c(Red)

- It is possible to construct an efficient mechanism in all of these examples, but rather than do this case by case, we will derive an umbrella mechanism that works in a whole range of cases.

General Framework

Return now to the general social choice setup.

- A society consisting of n individuals
- A set A of alternatives from which to choose.
- $v_i(x)$ is the value to i from alternative $x \in A$ being chosen.
- Monetary transfer scheme $t = (t_1, \ldots, t_n)$.

Thought Experiment

- Suppose for the moment that we know the value functions v_i of each individual i.
- We compute the utilitarian alternative x^*.
- Let's measure how much each individual i "contributes to the rest of society."

Thought Experiment

- First compute
$$\sum_{j \neq i} v_j(x^*)$$
- This is the total welfare of the society (not counting i).
- Next, let's ask how this would change if i were not a memer of society.
- We find the utilitarian alternative for the society which consists of all individuals *except* i.
- Call that x^*_{-i}. It will generally be different from x^*. We compute
$$\sum_{j \neq i} v_j(x^*_{-i})$$
- The difference
$$\sum_{j \neq i} v_j(x^*) - \sum_{j \neq i} v_j(x^*_{-i})$$
is a measure of how much i contributes to the rest of society. (It will often be negative, for example in the auction context.)

The Vickrey-Clarke-Groves Mechanism

We will construct a game in which player i receives a monetary transfer equal to the amount he contributes to the rest of society.

- The players are the members of society.
- The actions: each player will make a claim about his valuation function.
  - Recall that v_i is i's true valuation function.
  - So $v_i(x)$ is i's true value for alternative x.
  - Each player i will announce a valuation function $\hat{v}_i$.
  - The announcements are simulataneous.
  - So $\hat{v}_i(x)$ is i's stated valuation of alternative x.
  - She might announce $\hat{v}_i \neq v_i$, i.e. she might lie.
  - Since only she knows the true v_i there is no way to know whether she is telling the truth.
  - We need to give her the right incentives to tell the truth.

Outcomes

- When the players announce $\hat{v} = (\hat{v}_1, \hat{v}_2, \ldots, \hat{v}_n)$, the utilitarian alternative for $\hat{v}$ is enacted. Call it $x^*(\hat{v})$.
- Remember that the utilitarian alternative maximizes the sum of the (announced) valuations, i.e.

$$\sum_{j=1}^{n} \hat{v}_j(x^*(\hat{v})) \geq \sum_{j=1}^{n} \hat{v}_j(x)$$

  for any other alternative x.
- The last detail to specify is how monetary transfers are determined.

The VCG Transfer Rule

- Recall that in our notation $\hat{v}_{-i}$ refers to the list of announcements by everyone other than i.
- Let $x^*(\hat{v}_{-i})$ represent the utilitarian alternative for the society that excludes i.

$$\sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_{-i})) \geq \sum_{j \neq i} \hat{v}_j(x)$$

  for any other alternative x.
- In the VCG mechanism, when the list of announced valuation functions is $\hat{v}$, player i receives the transfer $t_i(\hat{v})$ defined as follows

$$t_i(\hat{v}) = \sum_{j \neq i} \hat{v}_j(x^*(\hat{v})) - \sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_{-i})).$$

The Vickrey Auction is a Special Case

Consider the simple problem of allocating a prize and apply the VCG transfer rule.

- If i reports the highest valuation,
  - then $x^*(\hat{v})$ = "give the prize to i"
  - and $x^*(\hat{v}_{-i})$ = "give the prize to the individual k with the second-highest value"

$$\sum_{j \neq i} \hat{v}_j(x^*(\hat{v})) - \sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_{-i})) = 0 - \hat{v}_k = -\hat{v}_k.$$

- If i does not report the highest valuation,
  - then $x^*(\hat{v})$ = "give the prize to the individual l with the highest value"
  - and $x^*(\hat{v}_{-i})$ = "give the prize to the individual l with the highest value"

$$\sum_{j \neq i} \hat{v}_j(x^*(\hat{v})) - \sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_{-i})) = \hat{v}_l - \hat{v}_l = 0.$$

The VCG is an Efficient Mechanism

- The VCG mechanism is defined not just for auctions but for any social choice problem.
- We will show that the VCG mechanism is efficient:
  1. All individuals have a dominant strategy to announce their true valuations.
  2. When they do so, the utilitarian alternative is enacted by the VCG mechanism.
- By construction the mechanism picks the utilitarian alternative for the *announced* valuations, i.e. $x^*(\hat{v})$. So once we show the first property, we will have that $\hat{v} = v$ and so $x^*(v)$ will be chosen, satisfying the second property.

Announcing Truthfully is a Dominant Strategy

- We need to show that announcing truthfully $\hat{v}_i = v_i$ is the best strategy no matter what the other individuals announce, i.e. no matter what $\hat{v}_{-i}$ is.
- If the others announce $\hat{v}_{-i}$ and i announces $\hat{v}_i$, i's utility is

$$v_i(x^*(\hat{v}_i, \hat{v}_{-i})) + t_i(\hat{v}_i, \hat{v}_{-i})$$

  we substitute the VCG transfer formula for t_i:

$$v_i(x^*(\hat{v}_i, \hat{v}_{-i})) + \sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_i, \hat{v}_{-i})) - \sum_{j \neq i} \hat{v}_j(x^*(\hat{v}_{-i})).$$

- Player i has to decide what $\hat{v}_i$ to announce. It will determine $x^*(\hat{v}_i, \hat{v}_{-i})$ but not $x^*(\hat{v}_{-i})$. So we can ignore the last term since it is unaffected by i's announcement.

Announcing Truthfully is a Dominant Strategy

- Suppose for the moment that i could choose the alternative x directly. What x would maximize

$$v_i(x) + \sum_{j \neq i} \hat{v}_j(x)$$

- The answer is $x = x^*(v_i, \hat{v}_{-i})$.
- But i cannot choose x directly, he can only choose $\hat{v}_i$ and then $x^*(\hat{v}_i, \hat{v}_{-i})$ will be chosen.
- Still, by announcing truthfully $\hat{v}_i = v_i$ he ensures that $x^*(v_i, \hat{v}_{-i})$ will be chosen.
- So announcing truthfully is the best thing he can do.

More Applications

Let's revisit the auction with externalities and compute the VCG transfers. Suppose the players report $\hat{v}$ and

- The efficient allocation is Z, i.e. $x^*(\hat{v}) = Z$. How much does z pay?
  - The first term in the formula $\sum_{j \neq z} v_j(Z) = -5$ because of the negative externality on y.
  - The second term, $\sum_{j \neq z} v_j(x^*(\hat{v}_{-z}))$ equals the second-highest value as usual.
  - Thus, according to the VCG rule z receives -5 minus the second-highest value.
- The efficient allocation is X, i.e. $x^*(\hat{v}) = X$. How much does x pay?
  - The first term in the formula $\sum_{j \neq z} v_j(X)$ equals zero.
  - So he receives 0 minus the second term, i.e. he pays the second term. The second term equals
    - ★ v_y if $x^*(\hat{v}_{-x}) = Y$.
    - ★ $v_z - 5$ if $x^*(\hat{v}_{-x}) = Z$.

More Applications

The designer dress example.

- An alternative is a specification of who wears which dress.
- Suppose that according to their announced valuations, they prefer opposite dresses, e.g,
  - Then for each individual i, $x^*(\hat{v}) = x^*(\hat{v}_{-i}$, so the payment is zero.
  - Idea: no conflict, no need for monetary payments.
- But if each announces that they prefer the same dress, then
  - The one announcing the higher value gets their preferred dress.
  - And pays the other's announced value.
  - Idea: when there is conflict it is resolved using a Vickrey auction.

The Espresso Machine

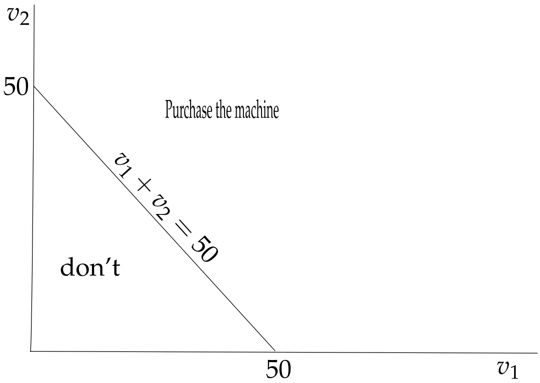
- Two roomates, with willingness to pay v_1, v_2 for an espresso machine
- The cost of the machine is \$50.
- We considered two mechanisms that were not efficient
  - Split the cost. (didnt achive the utilitarian solution)
  - Bargaining game. (no dominant strategies)

The VCG mechanism in the Espresso Machine Problem

- Lets apply the VCG mechanism.
- We must include the individual who owns the machine.
- His value for keeping the machine is 50.
- Suppose $\hat{v}_1 + \hat{v}_2 \geq 50$. but $\hat{v}_2 < 50$ and $\hat{v}_1 < 50$.
- VCG mechanism specifies that the machine should be purchased.
- VCG payments:
  - first term: $\sum_{j \neq 1} \hat{v}_j(x^*(\hat{v})) = \hat{v}_2$
  - second term:
    - Because $\hat{v}_2 < 50$, we get $x^*(\hat{v}_{-1})$ is not to buy the machine.
    - $\sum_{j \neq 1} \hat{v}_j(x^*(\hat{v}_{-1})) = 50$. (owner keeps machine)
    - So 1 receives $\hat{v}_2 - 50$, i.e. he pays $50 - \hat{v}_2$.
    - Likewise 2 pays $50 - \hat{v}_1$.
- What is the sum of the contributions from the two players?
- Answer: $50 - \hat{v}_1 + 50 - \hat{v}_2 = 100 - (\hat{v}_1 + \hat{v}_2)$.
- This is less than \$50.
- That is a problem.

Can We Do Better?

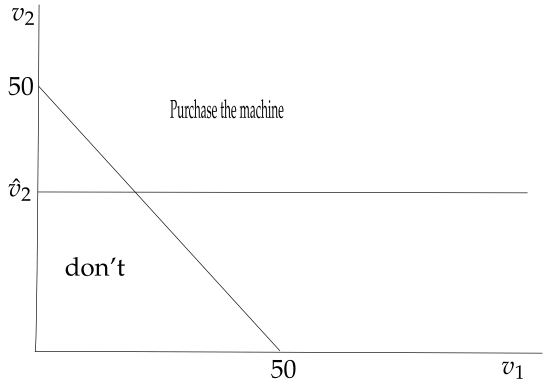
Is there any mechanism which is efficient and doesn't result in a deficit?



Recall the diagram for the utilitarian decision rule.

Can We Do Better?

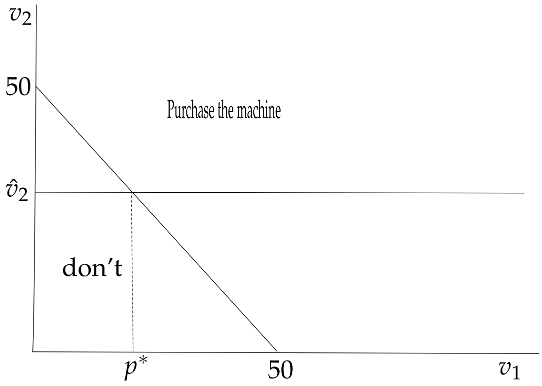
Is there any mechanism which is efficient and doesn't result in a deficit?



Suppose 2 announces willingness to pay $\hat{v}_2$. If the machine is purchased, how much should 1 be required to pay?

Can We Do Better?

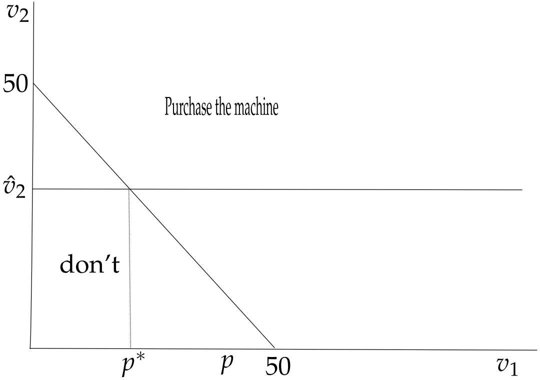
Is there any mechanism which is efficient and doesn't result in a deficit?



We will show that 1 should be required to pay p^*.

Can We Do Better?

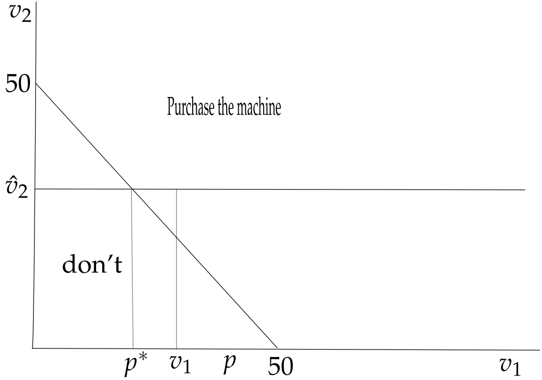
Is there any mechanism which is efficient and doesn't result in a deficit?



Suppose instead that the price was set at $p > p^*$.

Can We Do Better?

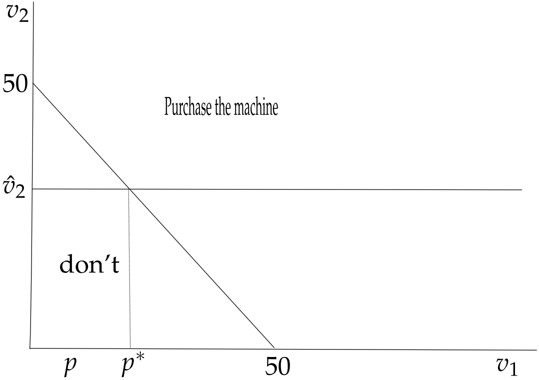
Is there any mechanism which is efficient and doesn't result in a deficit?



In this case 1 would have an incentive to lie when he has a willingness to pay v_1 that is between p^* and p. (He would want to understate his value.)

Can We Do Better?

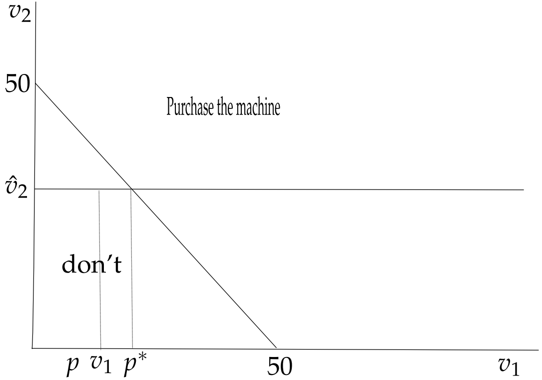
Is there any mechanism which is efficient and doesn't result in a deficit?



On the other hand, if the price were set below p^*, say at $p < p^*$, ...

Can We Do Better?

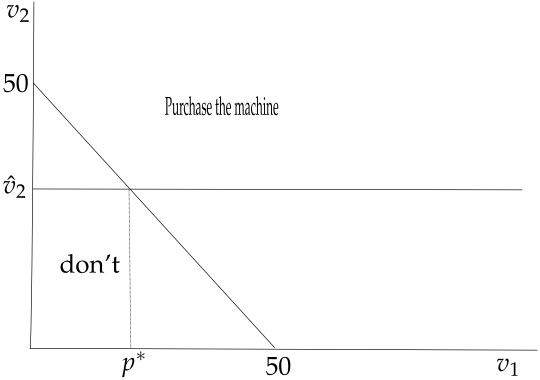
Is there any mechanism which is efficient and doesn't result in a deficit?



Then when 1's value is v_1, between p and p^*, 1 has an incentive to overstate his value.

Can We Do Better?

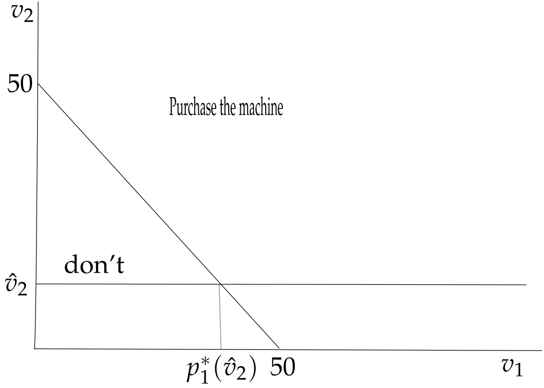
Is there any mechanism which is efficient and doesn't result in a deficit?



Thus, 1 must pay p^*. In this case, 1 will truthfully report his value, whatever it is.

Can We Do Better?

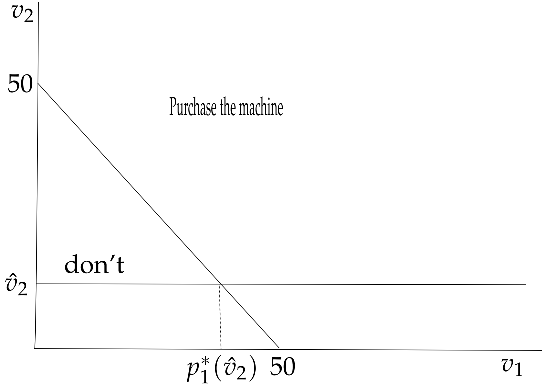
Is there any mechanism which is efficient and doesn't result in a deficit?



When we do this for all possible announcements $\hat{v}_2$ for player 2, we trace out the transfer rule for 1.

Can We Do Better?

Is there any mechanism which is efficient and doesn't result in a deficit?



This means that 1 always pays $50 - \hat{v}_2$. Exactly as in the VCG mechanism.

The VCG mechanism is the Only Efficient Mechanism

- Since the VCG mechanism is the only mechanism that
  - Makes truthtelling a dominant strategy
  - Implements the utilitarian rule
- And since the VCG mechanism yields a budget deficit,
- *There is no budget balanced, efficient mechanism for this social choice problem.*
- Ok then, the "first-best" is not attainable. What's the best we can do with a budget-balanced mechanism? (The "second-best.")